CHAPTER 14

DYNAMIC BEHAVIOR OF DISEQUILIBRIUM RETAIL MODELS: CATASTROPH AND BIFURCATION

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14.1 INTRODUCTION

Urban redevelopment and metropolitan restructuring come to be discussed as the most important issue of regional planning in Japan. Considering resent growth of Tertiary Industries, and in order to utilize the existing urban infrastructures which has been constructed in relation to the development of commercial cores, analysis of dynamic behavior of the retail and service activities is very important for regional planning. To date, many types of retail location model were developed for analysis of spatial formation of service activities and competition between them.

These models are typically based on the demand-oriented growing process. However, even commercial cores with plenty of neighbor consumers that are suffering from the decrease in sales are sometime observed, such as newly developed residential areas or old fashioned shopping street. As their service style doesn't catch up the swift change of consumer's appetite, increase of demand does not yield growth in sales and activity size. Before the same demand condition, either growth or decay can exist. Plenty of demand does not always assure a growth of activities, that seems to be an serious limit of the demand-oriented approach and models to date. It may be quantitative and qualitative disparity between demand and supply at every instance that determines the locational change of service activities. Dynamic retail model originally developed by Harris and Wilson (1978) is one of the disequilibrium models which have much advantage to analyze these mechanism.

Dynamic behavior of model is inevitably determined to some extent when we specify the model structure; for example, that of the Harris-Wilson Model, which takes the form of non-linear differential equations, is known to be determined by a few key parameters. Information about dynamic behavior caused by parameter shift is very valuable to detect the range of parameters where the model is structurally stable and then comparative statistics analysis is applicable; that is also helpful to discuss the conditions of structural change and their rich implications for policy making.

This paper aims to discuss the way of numerical analysis of dynamic behavior of the model and to assess the applicability for policy analysis. Needless to say, parameter estimating method is also very important issue to examine the status of real world system and to make effective policies. Because the authors have already discussed the problem in other paper, this paper will be focused on the way of dynamic behavior analysis.

The remainder of this paper consists of five parts. In the next section, studies on dynamic behavior of regional systems are reviewed to seize the essence of the findings and waiting's. The way of numerical analysis is discussed in Section 14.3. Because general solution of the multi-centers competing system cannot be obtained, we focus our discussion into simple situation where only two major centers exist and suggest the way of aggregation (Section 14.4). Section 14.5 is prepared to assess the applicability to policy analysis. The last part conclude our discussion.

14.2 STUDIES ON DYNAMIC BEHAVIOR OF REGIONAL SYSTEMS TO DATE

Dynamic behavior of regional systems is one of the most interesting targets of the regional scientists. However, lack of adequate theory or tool dealing the complicated dynamic behavior prevented a theoretical or sophisticated approach, rules are only hypothesized from historical observations. Catastrophe Theory by Thom was the first outstanding framework for analysis of complicated dynamical systems, such as social systems or biological systems. Since the late 1970's, geographers came to attack the behavioral analysis of regional system with the aid of catastrophe theory; Wilson and his colleagues left many pioneering works. In early 80's, the approach became more theoretical and sophisticated focusing on uniqueness or stability analysis. Due to the difficulties for manipulation, those studies were limited in two centers problem, however, many interesting findings were obtained. In parallel to these streams, behavior of multi-zonal system were discussed by the aid of Synergetics, which dealt the self-organization caused by micro-macro interaction. Here, we give a glance at these works and seize the essence of them.

The pioneering applications of catastrophe theory to regional modeling owe much to A. G. Wilson and his colleagues. In the paper of Poston and Wilson (1977), they showed the optimal solution of the facility allocation model yields Fold Catastrophe, that was the first application of catastrophe theory in the literature of regional science. Harris and Wilson (1978) formulated the dynamic retail location model, where change of activity level is determined by the profit in each commercial center; the sales is calculated by the single constrained gravity model, and running cost is derived from the present activity size. In this paper, uniqueness and stability of equilibrium were discussed at first time, jump of equilibria caused by parameter shift was also revealed. Following to this epoch making paper, they continued the studies of dynamic gravity models to show that the behavior of them are classified to Fold Catastrophe.

Other researcher also developed the way of dynamic analysis. As stated before, these theoretical analyses were limited to the simple case such as two centers competes. Study on the uniqueness of equilibrium and number of equilibria was accelerated. For example, Rijk, Vorst (1983) analyzed the equilibria of the spatial interaction model using the Poincare-Hopf's Index Theorem or Brouwer's Fixed Point Theorem, number of equilibria was related to the parameter of size attractiveness. Chudzynska, Stodkowski (1984) derived the satisfying condition on size attractiveness for uniqueness of equilibrium of gravity model. Kaashoek, Vorst (1984) showed that spatial interaction model gives a Cusp Catastrophe, and numerically calculated the bifurcation set for simple case; transportation cost decides the possibility of coexistence of two centers. This paper takes similar approach as them, but more generalized. Kohsaka (1986) formulated the two centers competition as Lotka-Volterra Model and derived the coexistent condition.

The third stream is the studies on dynamic behavior of urban system with many zones. Allen, Sanglier (1978) modeled the interaction between population and regional employment as the dynamic Central Place System. Based on the simulation experiments, they showed that the change in network are resulted in structural change of the regional system. Beaumont, Clarke and Wilson (1981) numerically simulated the urban growth pattern using gravity model.

The important findings of these studies are summed up as follows; structural change of the regional system can be well expressed using nonlinear differential equations. Although analytical solution is not available, essence of the solution's behavior can be speculated based on the configuration of equilibria. Number or location of equilibria can be suddenly changed by parameter shift. As discussed in the next section, our approach much owes to these findings.

On the other hand, the opening issue is summarized as the more quantitative discussion in the context of model applicability or credibility for operational use. Such information as concrete site of the bifurcation set is very worthy when we check the applicability of the model.

14.3 THE WAY TO ANALYZE DYNAMIC BEHAVIOR OF HARRIS-WILSON MODEL

In this study we consider the typical dynamic retail model originally formulated by Harris and Wilson (1978), the simple formulation is as follows:

$$dW_j = e(\sum_i O_i \frac{\exp(a\,W_j - C_{ij})}{\exp(a\,W_l - C_{il})} - k\,W_j) \tag{1}$$

where,
O_i stands for the demand in zone i, W_j is the size of service center j measured by the number of employee in service sector, C_{ij} is the transportation cost between zone i to center j, a is unknown parameter of attractiveness of size, k is unknown parameter of supplying potential measured by sales per employee, and e is unknown parameter of adjustment speed. The first term in the parenthesized part in the right hand side means the demand in the market area of each service center, the second one is potential to supply at the same center. Therefore, this model can be grasped as the model of retailer's adjustment process for the dis-equilibrium between demand and supply. It is very difficult to get the analytical solution of this type of model. However, we can estimate the macro behavior of the system as follows. As time goes by, every state will converge to one of the few stable singular points, called as attractor or sink.

Each sink have the area called as catchment basin, such that the state in the area will converge to the sink in long run. Every state will be converged to the sink correspond to the catchment basin containing the state. As illustrated afterwards, the frontier between the catchment basins are closely dependent to the location of the unstable singular points, i.e. source and saddle. The singular points are determined as the point that no change will occur. Because such condition is only true when demand equals to supply, these singular points are no more than equilibria states. As shown in the preceding studies, configuration of these equilibria is known to be determined by few key parameters in the model. Now, let us consider the parameter space, where, every point corresponds to one differential equation. We can divide the parameter space into some sub-spaces where the number and topological configuration of equilibria are same. Frontier between such sub-spaces called as bifurcation set. In order to estimate the configuration of equilibria, the shape of bifurcation sets is important as well as the changing pattern of equilibria when parameter shifts across each of the bifurcation sets.

Figure 14.1 shows the vector field of two centers problem for some value of parameters. Size of each center is plotted on the vertical and horizontal axis, direction of change calculated by the model is shown as the flow lines. In this case there is five equilibria states; three of them are stable (sink), and the other two unstable saddle points are sited between them. As times goes by, every initial state approaches to the constant-sum valley line and converges into one of the stable equilibria at last. The space is divided into the three catchment basins around each stable sink by two strait border lines meaning constant-disparity condition, which are ridge shaped in the vector field. The two unstable equilibria (Saddle) are stated at the intersection of the valley line and the ridge lines, respectively; this relation implies that the unstable equilibria determine the configuration of catchment basins.

14.4 BIFURCATION SETS AND BEHAVIOR OF TWO CENTERS PROBLEM

When there are only two centers, equation (1) will be rewritten as follows;

$$dW_1 = e(O_1 \frac{1}{1+\exp\{aW_2 - aW_1 - (C_{12} - C_{11})\}} + O_2 \frac{1}{1+\exp\{aW_2 - aW_1 - (C_{21} - C_{22})\}} - k\, W_1) \tag{2}$$

Here we define the state variables $x = kW_1$. Ordinally, inter-zonal transportation cost is larger than intra-zonal one, here the difference of them ($C_{12} - C_{11}$ and $C_{21} - C_{22}$) are rewritten as parameter C, which means relative inter-zonal transportation cost to intra-zonal transportation. If inter-zonal trunk road is improved, value of C will decrease, and improvement of local network will increase C. In equilibrium, total demand and total supply is identical, which is written by $O = O_1 + O_2$. X is total activity size fill the total demand in the region. Considering the relation that a $X = kO$, increase of regional population and increase of consumption per capita make lager value of X. Market size of each zone can be given using O and r, as follows:

$$O_1 = r\, O \tag{3}$$

$$O_2 = (1-r)O \quad (0 \leqq r \leqq 1) \tag{4}$$

where, r is consumer disparity ratio, equality gives value of 0.5. Equation (2) take the form as,

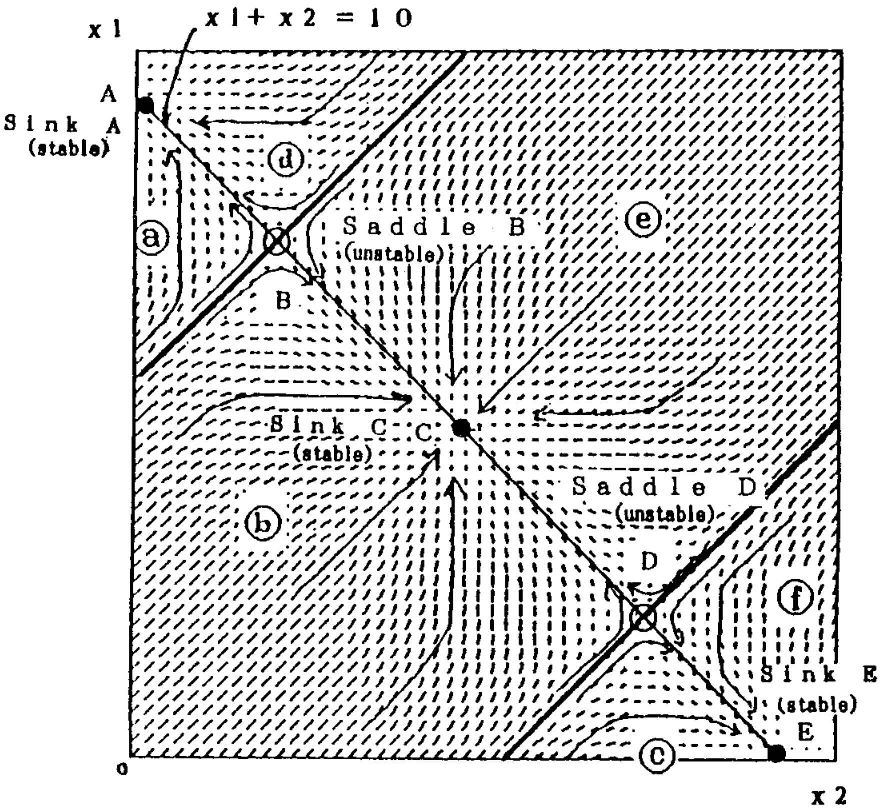
$$dx = eaO(\frac{r}{1+\exp(X - C - 2x)} + \frac{1-r}{1+\exp(X + C - 2x)} - \frac{x}{X}) \tag{5}$$

Here, W_2 is derived as $(X - x)/k$, we consider only the change of x. Because e, a, O are positive, we must only check the sign of the parenthesized part in the right hand side, as to discuss the number or stability of equilibria. The three terms in the parenthesis mean intra-zonal demand, inter-zonal demand, and supply of center 1, respectively. Therefore, this part is rewritten as excess-demand function, $h(x)$, as follows:

$$h(x) = hd(x) - hs(x) \tag{6}$$

$$hd = \frac{r}{1+\exp(X - C - 2x)} + \frac{1-r}{1+\exp(X + C - 2x)} \tag{7}$$

$$hs = \frac{x}{X} \tag{8}$$



$\alpha = 0.1$ $\kappa = 20$
$O1 = 1000$ $O2 = 1000$
$C11 = 5$ $C12 = 10$
$C21 = 10$ $C22 = 5$

ⓐⓓ Catchment Basin of Sink A
ⓑⓔ Catchment Basin of Sink C
ⓒⓕ Catchment Basin of Sink E

Figure 14.1: Vector field and equilibria of two centers competition

While $hd(x)$ is larger than $hs(x)$, dx is positive and x increases. The contrary condition gives the decrease of x. At the edges of defined space of x, i.e. $x = 0$ and $x = X$, $h(x)$ gives the different sign as follows:

$$h(0) = \frac{r}{1+\exp(X-C)} + \frac{1-r}{1+\exp(X+C)} > 0 \tag{9}$$

$$\begin{aligned} h(X) &= \frac{r}{1+\exp(-X-C)} + \frac{1-r}{1+\exp(-X+C)} - 1 \\ &< r + (1-r) - 1 = 0 \end{aligned} \tag{10}$$

Therefore, at least one stable equilibrium exists for any value of X, C, r.

Here, $hd(x)$ is continuous and monotonic increasing function of x, with only two inflection points, $hs(x)$ is increasing linear function.The intersections of $hd(x)$ and $hs(x)$, giving the equilibria, can exist at 1, 3, or 5 points unless they are multiple solutions.

Now, we check the number of equilibria. For many value's combination of parameter X, C, r, the number of equilibria is numerically calculated. Figure 14.2 shows the three dimensional parameter space divided into the four sub-spaces according to the number of equilibria, here, three equilibria case is divided into two cases by possibility for standing together.

[a] This sub-space gives unique stable equilibrium, every state approaches to that state at last.

[b] Three equilibria: two are stable and the other is unstable. One stable equilibria means that two centers stand together, and the other means the centralization to the zone with the larger hinter market. As catchment basin of the former is larger than the one of the latter, two center will stand together at last.

[c] Five equilibria: three are stable, two unstable between them. One stable equilibria means that two centers stand together, and the others mean the centralization to each of center, respectively.

[d] Three equilibria: two stable centralizing equilibria and an unstable one dividing the catchment basin of them.

Equation of bifurcation set are also calculated. First we find the value of parameter where the number of equilibria change, by halving search. The twenty such points are well regressed by simple functions, which are also shown in Figure 14.2.

Shift pattern of equilibria caused by crossing the bifurcation sets is also important information. Various transition patterns are classified into five groups, considering the location of the bifurcation sets showed in Figure 14.3. In this figure, vertical axis is plotted as the size of center 1 proportional to the total size $X(x/X)$; smooth line indicates a stable equilibrium and dotted one does the unstable equilibrium.

[Case 1: Crossing the Bifurcation Set 1]

Crossing the Bifurcation set 1, transition between sub-spaces [a] and [d]

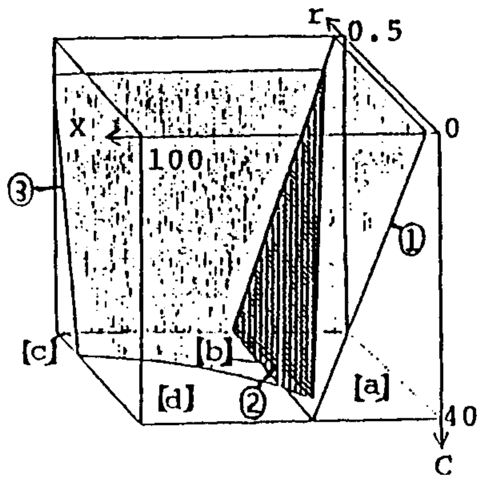
Number of Equilibria

[a] 1

[b] 3 (One of the Two Stable Equilibria means that two centers stand together)

[c] 5

[d] 3 (Each of the Two Stable Equilibria means much disparity between two centers)



① $1.08C-X+2.60=0$ (11)

$(n=20\ ,\quad r=0.9989)$

② $1.28C-1.23X+6.88r+1=0$ (12)

$(n=20\ ,\quad r=0.9998)$

③ $1.70C-1.78X-13.44r+3.51Xr-1=0$ (13)

$(n=20\ ,\quad r=0.9998)$

n: number of data

r: multiple correaction coefficients

Figure 14.2: Number of equilibria and bifurcation set (Two centers competition)

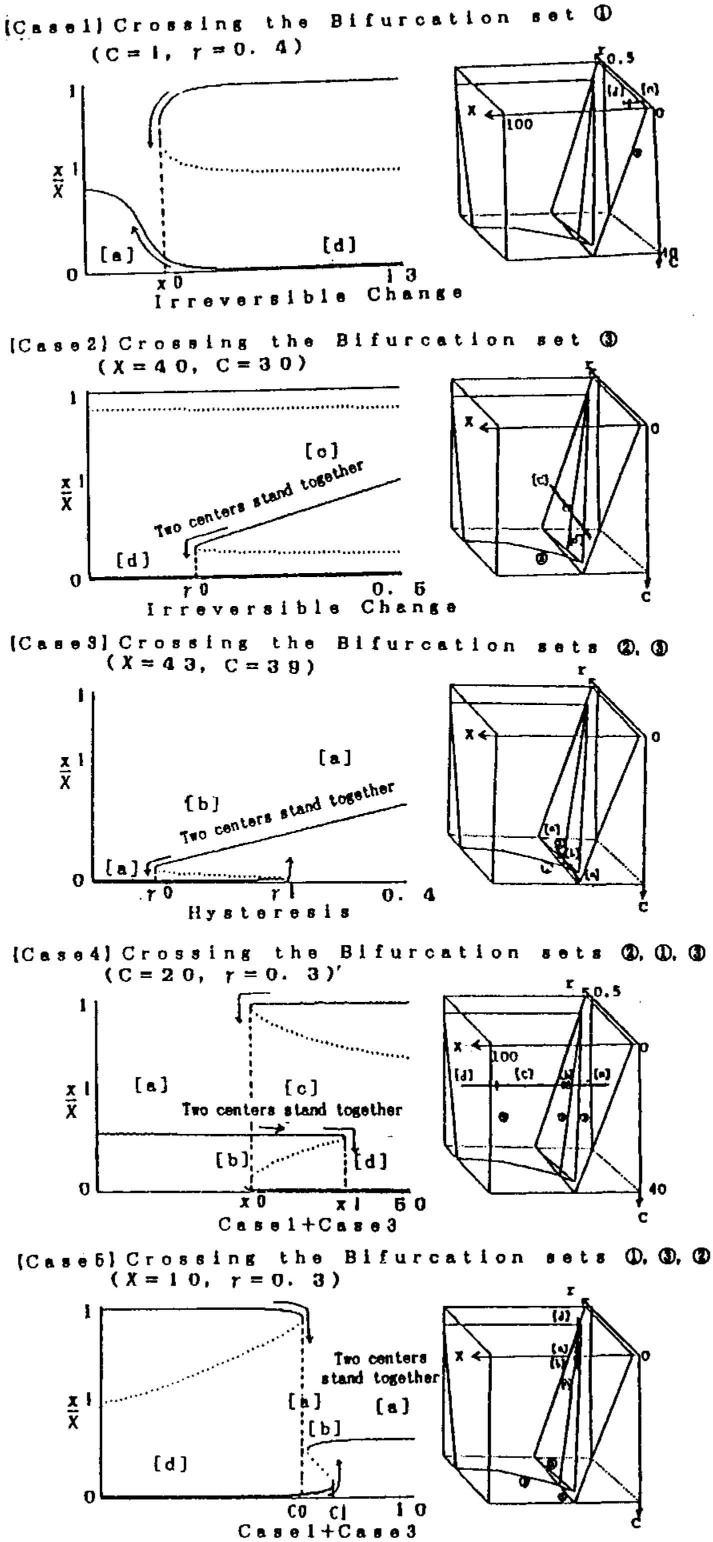


Figure 14.3: Shift of equilibria caused by crossing the bifurcation sets

occurs, when r=0.4, C=1, and parameter X shifts, for example. The top figure shows the Z-shaped curve. While X is smaller than the critical value, $X0$, all equilibria shifts smoothly as the decrease of parameter X. At the point of $X0$, the stable equilibrium meaning centralization to the center with smaller hinter market suddenly disappear. After this type of change, the recovery of X does not yield any more sudden change. Therefore, this structural change is irreversible one, that is very typical for non-linear dynamic system. The same change can be seen when parameter C or r shifts across the bifurcation set 1.

[Case 2: Crossing the Bifurcation Set 3]
When c=30, X=40, shift of parameter r yields the irreversible structural change. The stable equilibrium for coexistence of the two center disappear at the point of critical value $r0$. Crossing by shift of X also give the same result.

[Case 3: Crossing the Bifurcation Sets 2 and 3]
Then parameter shifts across not only one bifurcation set, other type of structural change occurs. The third figure shows the S-shaped curve when C=39, X=40, and parameter r shifts across the sub-spaces [a], [d], [a]. As r decrease from the value of 0.4, the unique stable equilibrium shifts smoothly until the critical value $r0$, under which only centralization equilibrium is stable. If r increases again, no change occurs at the point of $r0$, but at the other critical point $r1$ the centralization equilibrium suddenly disappear. Although both increase and decrease give sudden change, critical points are different from the direction of shift. This nature is also popular for non-linear system called as hysteresis. It is needless to say that other parameter's shift across the same sets gives the same change.

[Case 4: Crossing the Bifurcation Sets 2, 1, 3]
When X shifts at the condition of r=0.3 and C=20, three crossing occur as shown in the forth figure. This complicated transition between sub-spaces [a], [b], [c], [d] can be seen as the duplication of Irreversible Change (Case 1) and Hysteresis (Case 3).

[Case 5: Crossing the Bifurcation Sets 1, 3, 2]
Other type of duplicated change is shown in the bottom, which shows the case of X=10, r=0.3. Sequence of crossing is differ from the last case. Parameters shift in sub-spaces [d], [a], [b], [a]. Like Case 4, this case is also derived as duplication of Z-shaped curve (Case 1) and S-shaped one (Case 3).

As shown here, parameter shift crossing the bifurcation sets will suddenly alter the number and location of equilibria, resulted in irreversible or hysteresis phenomena.

In the field of astromony for example, dynamical system which consists of more than two objects was attacked to resolve. However, even analytical solution of three objects problem is not revealed to be available. Just like this situation, it is pessimistic to get general findings about multi-center problem. In this section, therefore, we only consider the special case of three centers problem such that two centers are much larger than the third one. It is not so surprising that we approximately convert such system into two centers system. Here, we try to check the applicability of the findings about two centers by comparing the following three solutions;

(1). Two centers problem neglecting the third center,

(2). Three centers problem under the assumption that the third center is much smaller than the others, ($W = 0$).

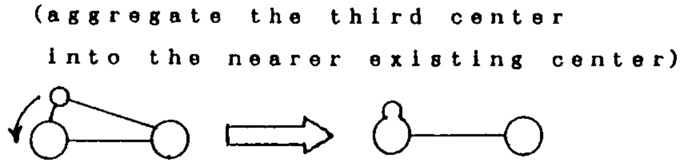
(3). Two center problem derived from aggregation of the third center into the first or second one.

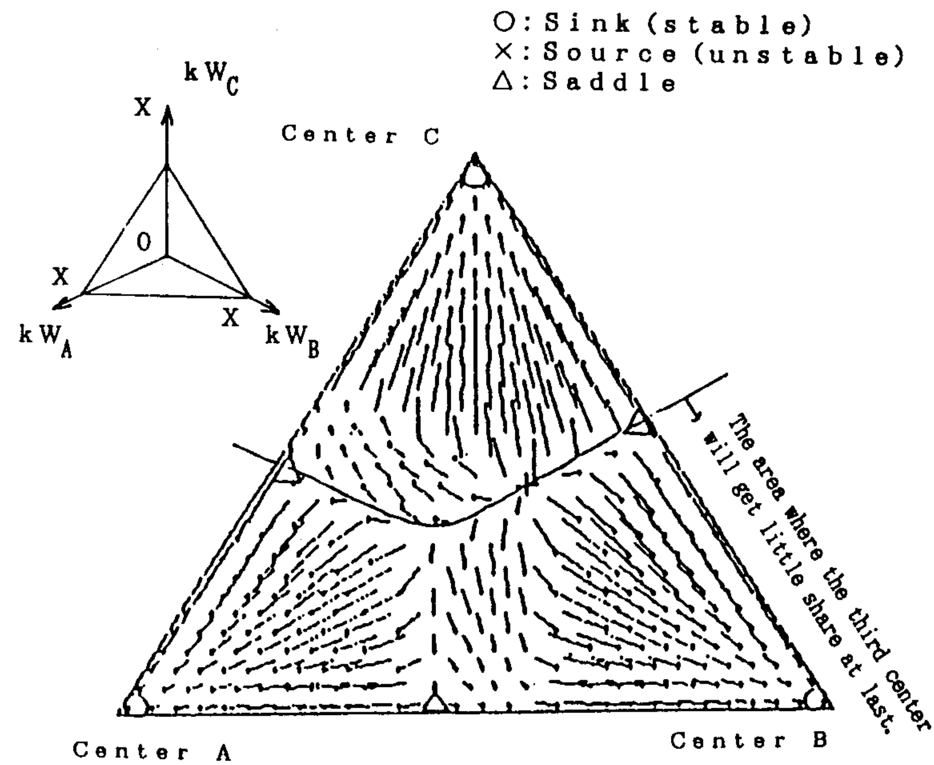
Figure 14.4 shows the case where the third center is considered other than the two existing centers of same size. Middle figure shows the vector field on the constant-sum plane in the three dimensional space. There are three stable equilibria which mean centralization to one of each centers respectively. On the border between three catchment basins, one source and three saddles exist. Unless the initial size of the third center is so large as is in the catchment basin of the centralization of the third center, either the first or second center will grow up at last. Then, the assumption that the size of the third center is very smaller than the others is true in that occasion, and the second way is reliable, which tells us there is two stable equilibria centralization to either the first center or the second. By the first way neglecting the existence of the third center, however, different result is obtained, i.e. there were another stable equilibrium where the first and the second center might stand together.

As illustrated here, existence of the third center has some effects on the competition between the two existing centers such as change in number of equilibria, which cannot be neglected for the purpose of structural analysis. The equilibria are also defined as the points where the excess-demand equals to zero, for two centers competition, that was written in eq. (6) as before. On the condition that the size of the third center equals zero, we can also define the similar excess demand function for the three centers problem. The bottom figure compares the excess-demand curves for the three methods. Very little difference between the curve of three centers problem and that of the aggregated two centers. From this coincidence, applicability of the third method, aggregation into two centers, can be concluded. Unnatural aggregation, such as adding into the further center, however gives bad approximation.

Figure 14.5 shows the case when we consider the little third center other than the two existing centers of different size. In this vector field, there are three stable equilibria and two unstable saddles. The catchment basin of the third center is also very small in this case. According to the excess demands curves, neglecting of the third center yields to estimate an nonexistent stable equilibria, on the other hand the aggregation gives a better approximation of the system. Unnatural aggregation into the smaller center results in worse approximation than the neglecting.

From the analysis of this section, we can concluded as follows; we cannot neglect the existence of the third center even if the center is





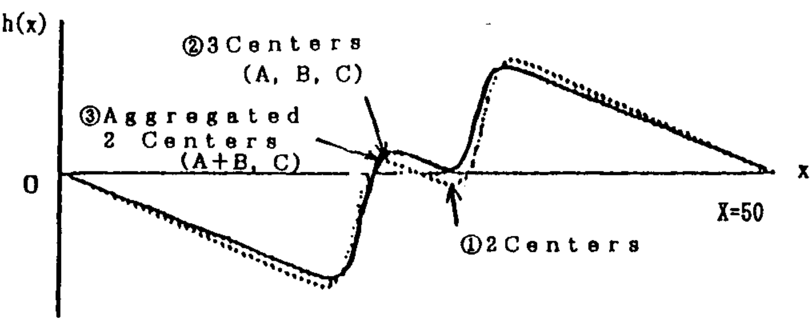


Figure 14.4: Consider the third center other than the two existing centers of same size.

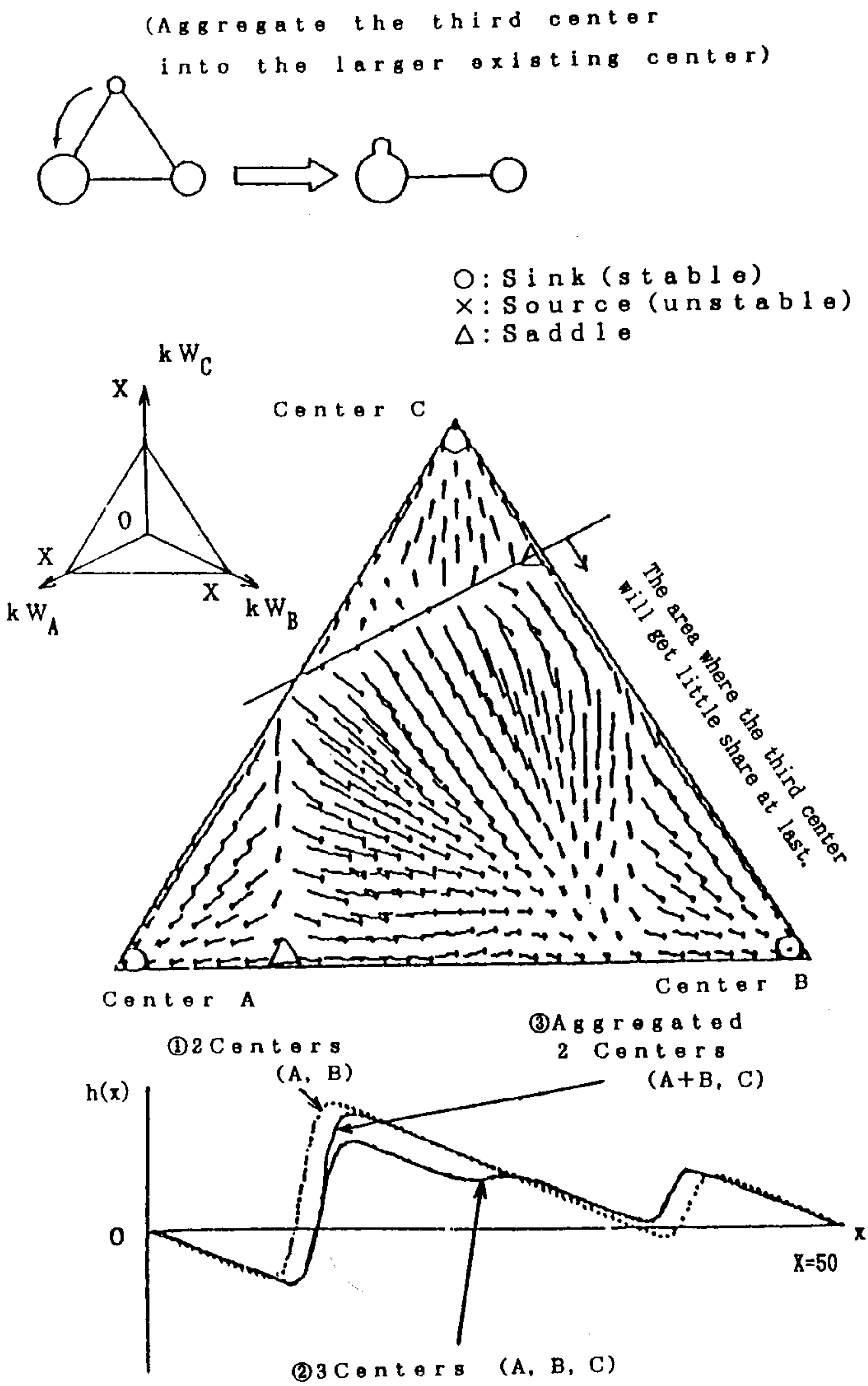


Figure 14.5: Consider the third center other than the two existing centers of different size.

small in size, and good approximation is obtained by aggregating the third center into the nearer or larger existing center and solving the aggregated two centers problem. By repeating this aggregation procedure, the multi-center problem such that only two major centers exist can be well approximated by aggregated two centers problem.

14.5 APPLICATION TO POLICY ANALYSIS

This section discusses the applicability of the information derived from our analysis, as stated, previous part to policy analysis. Here, we consider the newly developed residential area located in the suburbs. It is sometime the case that only the commercial core in the city center gathers consumers and that no growth of new commercial core requires inhabitants long trip for shopping. Therefore, raising new commercial core in the newly developed suburban residential area may be one of the aims of regional policy. If newly residential zone has so much population as the old zone (parameter r = 0.3 - 0.5) and there is not so much agglomeration of commercial activities ($x < 0.3X$), the status may be either [b], [c], or [d].

The first policy may be to allocate some size of new commercial facilities compulsory. The size must be determined as to exceed the frontier of catchment basin or unstable equilibrium, but deficit or other difficulties are unavoidable in long time. Then we had better combine with other development policies such as transportation infrastructure development, to change the regional structure as to ease the difficulties of growth. As expressed in the following, the effective policy is very differ from the present status of the region.

When the region is in the state of [b], increase of C across the bifurcation set 2, by improvement of intra-zonal transportation condition, yields the transition to the state [a], where new center can stand with the older one. Steady increase of the population, with the result of increase of r, also gives the state [a]. Because there is no effective policy to shift parameter X, these two ways must be discussed.

In the case that the present status is [c], the similar improvement of intra-zonal transportation is effective. Because hysteresis is seen, patient execution of the policy is important. Different from the last case, increase of population yields no structural change.

When the present status is [d], parameter must be shifted across the three bifurcation sets. Increase of C does not always have good effect. In this case, the policy must be discussed in stage wised; to shift into [c] is first step, transition into [a] is the next.

It is very difficult to discuss the timing and quantity of infrastructure development, however the result of our structural analysis must be helpful for make effective strategy.

14.6 CONCLUSION

We have analyzed the dynamic behavior of the Harris Wilson Model, a typical dynamic retail model, in more quantitative way than the previous studies. In the systems of non-linear differential equations, configuration of equilibria determines the total dynamic structure of the system. Because a few key parameters play an essential role in deciding this configuration of equilibrium states, the discussion was concentrated to the relation between them using numerical calculation. In our model, three key parameters were derived for two centers problem, i.e. inter-zonal transport condition, total demand size, and disparity in market size. It is our worthy findings that we estimated the function of bifurcation set in the parameter space, as well as the transition pattern crossing them. Based upon them, dynamic stability and applicability of the model can be checked. Although our study was only in the case of two centers competition, basic findings are also in the case for some of multi-central problem, the way of aggregation was also suggested. Applicability for actual policy making is not resolved yet, however combination with the parameter estimating method, it has comes to be discussed soon.

One of the interesting future tasks is to consider the two interacting disequilibrium markets, such as market of goods and labour market in the region. Equilibrium of one market need not to stand together with either of equilibria of the other market. Dynamic orbital of the regional development may be determined by such structural combination between regional markets.

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